Skyrmion dynamics on the unstable manifold and the nucleon-nucleon interaction

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Abstract

The unstable manifold of the $B = 2$ sector of the Skyrme model is constructed numerically using the gradient-flow method. Following paths of steepest descent from the $B = 2$ hedgehog, we apply a collective coordinate description for the motion on the manifold to extract a Hamiltonian, approximately valid for the quantum description of the low-energy nucleon-nucleon interaction. The resulting potential – obtained in the Born-Oppenheimer approximation – is in qualitative agreement with phenomenological potentials.

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Because of the non-abelian character of QCD, a first-principles derivation of low-energy hadronic physics has proven exceedingly difficult. Based on the large-N_c limit, where N_c is the number of colors [1,2] an interesting non-perturbative approach is offered by effective chiral meson models, as witnessed by their great phenomenological success. A popular realization is the Skyrme model – rather simple in nature yet remarkably realistic [3]. Its virtue lies in the fact that the non-linear classical field equations allow for topological solitonic solutions, called skyrmions, which carry baryon number, B [4], thus unifying the mesonic and baryonic sector in a common description. Aside from single-baryon properties, the multibaryon problem is of great significance in an attempt to understand complex nuclei within this framework. This poses a difficult problem since the starting point is inherently classical and 'requantization' is highly non-trivial. Most promising, in this respect, is the $B = 2$ sector, especially the description of nucleon-nucleon scattering, which will be the focus of the present work.

The Skyrme model is specified by the Lagrange density

$$\mathcal{L} = \frac{f_\pi^2}{4} Tr[\partial_\mu U \partial^\mu U^\dagger] + \frac{1}{32e_s^2} Tr([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U])^2 + \frac{m_\pi^2 f_\pi^2}{2} Tr[U - 1], \tag{1}$$

where $U \epsilon SU(2)$ denotes the non-linear pion field. The first term is the familiar non-linear sigma model, while the second is one of the possible four-derivative terms, derived systematically in chiral perturbation theory. A representation of the U-matrix is of the form $U = (\sigma + i\vec{\tau}\vec{\pi})/f_\pi$, with $\vec{\tau}$ being the three Pauli matrices. The parameters of the model are the pion decay constant f_π (=93 MeV), the Skyrme parameter e_s (=4.76) , related to the $\rho\pi\pi$-coupling constant [5] and the pion mass m_π (=138 MeV). The values quoted in brackets, which will be used in our calculations, ensure the correct asymptotic behavior of the fields, rendering the correct one-pion exchange tail of the nucleon-nucleon potential [6].

In order to assess the applicability of the Skyrme model beyond the $B = 1$ sector, where one gets reasonable results for the properties of the nucleon [7], a study of the $B = 2$ sector should yield an acceptable description of nucleon-nucleon scattering as well as the deuteron bound-state. Starting from classical skyrmion-skyrmion interactions one has to decide on

the appropriate field configurations in the $B = 2$ sector. Since the Skyrme-Lagrangian (1) is viewed as a model for large-N_c QCD, the semi-classical quantization procedure demands that we start from the minimal-energy configurations. As has been argued by Manton [8], the appropriate procedure to exclude undesirable vibrational modes, is to calculate the classical fields on the so-called 'unstable manifold' of $B = 2$ configurations. In this respect, an important role is played by the $B = 2$ 'hedgehog', which is of the form $U = e^{i\vec{\tau}\vec{\pi}F(r)}$, characterized by the 'profile function' $F(r)$ with boundary conditions: $F(0) = 2\pi$ and $F(\infty) = 0$. This special configuration is a saddle point on the hyperplane of minimal energy configurations and can therefore serve as a 'source' for further, less energetic $B = 2$ configurations. A complete class of configurations can be obtained numerically via the gradient-flow method [9]. One evolves the fields along a path of steepest descent, originating from the B=2 hedgehog which is initially excited by an unstable mode. The set of all decay paths defines a twelve-dimensional unstable manifold, called M^{12}, to be identified as the manifold of minimal-energy configurations in the $B = 2$ sector. The unstable modes of the $B = 2$ hedgehog have been reported in [10] by treating the hedgehog on compactified spatial 'hypersphere' and have been recalculated in [11] in 'flat' space, using lattice discretization. In the latter work the gradient-flow method and its numerical implementation was also developed in detail.

The calculations in [11] were performed on a three-dimensional grid using a finite differencing scheme for solving the gradient-flow equation. Applying a collective-coordinate description to the motion on M^{12}, which is always possible locally, the gradient-flow equation in collective coordinate space reads as

$$M_{ij}(Q)\frac{\partial Q_j}{\partial \tilde{t}} = -\frac{\partial V(Q)}{\partial Q_i}. \tag{2}$$

The evolution parameter $\tilde{t}$ is a pseudo-time, having a dimension (time)^2. The matrix $M_{ij}(Q)$ is the metric- or mass tensor on M^{12}, while $V(Q)$ denotes the interaction potential between two skyrmions. Both quantities depend, in general, on three relative collective coordinates of the $B = 2$ system. For well-separated skyrmions, these coordinates can be defined as

the distance R and the relative isospin orientation C between the skyrmions. Taking the separation axis – in the body-fixed frame – along the z-direction, the matrix C can be written in terms of two Euler angles β and χ as [6]

$$C = \cos(\frac{\beta}{2})\cos(\frac{\chi}{2}) + i\tau_2\sin(\frac{\beta}{2}) + i\tau_3\cos(\frac{\beta}{2})\sin(\frac{\chi}{2}). \tag{3}$$

In addition to the three relative coordinates, a $B = 2$ configuration is characterized by nine global collective coordinates, which fix the position of the center of mass and the overall orientation in space and isospace. Subsequently, we shall treat these coordinates in the adiabatic approximation, since they describe the 'zero-mode motion' of the skyrmions.

By exciting the $B = 2$ hedgehog with different linear combinations of the elementary unstable modes, one can – in principle – construct the entire unstable manifold numerically. Denoting the six unstable modes of the $B = 2$ hedgehog by $\delta U^{(M,i)}$ (magnetic mode [10]) and $\delta U^{(E,i)}$ (electric mode [10]), where $i = x, y, z$ specifies the symmetry axis of the unstable mode (for details see [11]), a general combination of the unstable modes, in the body-fixed frame, can be written as

$$\delta U = (\cos\varphi)\delta U^{(M,z)} + (\sin\varphi)\left[(\cos\theta)\delta U^{(E,z)} + (\sin\theta)\delta U^{(E,x)}\right]. \tag{4}$$

All other combinations of the unstable modes are related to one of the combinations above plus a global transformation of the $B = 2$ hedgehog. Thus, in the body-fixed frame, only the two mixing angles φ and θ are of importance. Together with the pseudo-time parameter $\tilde{t}$, these are three collective coordinates which uniquely specify the field configurations along the paths of steepest descent. As discussed above, another complete set of collective coordinates is given by the three relative collective coordinates R and C. In numerical calculations the latter are easily extractable only for well-separated skyrmions.

Using the finite-difference method as in [11], we calculate the paths of steepest descent from the $B = 2$ hedgehog numerically. Several paths are shown in Fig. 1 and Fig. 2, where the interaction potential along a given path is displayed as a function of the distance R between two skyrmions. For the distance coordinate we use a quadrupole definition [11].

The computational details can be found in [12]. Depending on the initial mixing angles in (4), the excited $B = 2$ hedgehog, positioned at the maximum of the interaction potential, can either split into two distinct $B = 1$ skyrmions (Fig. 1), or the paths end in the field configuration of minimal energy – the well known torus configuration (Fig. 2). The latter is classically bound by 120 MeV and is a source for large attraction in the interaction potential [6]. The path of steepest descent which results from the gradient-flow of two well-separated attractive skyrmions ($\beta = \pi$) is also shown in Fig. 2. The field configurations along this path can be described entirely by the relative distance coordinate which, together with the nine global collective coordinates, lead to a ten-dimensional submanifold of M^{12}, usually called M^{10} [8,13,14].

Figs. 1 and 2 indicate that the motion on the unstable manifold is generally quite complicated. The skyrmions move relative to each other in R and also tend to rotate in C whenever necessary in order to approach a configuration with less energy. As discussed in [11], we can identify the relative isospin orientation C between skyrmions for three different paths of steepest descent on M^{12}. Along the two elementary decay paths from the $B = 2$ hedgehog for excitations with the unstable modes superimposed by mixing angles of ($\varphi = 0$) and ($\varphi = \frac{\pi}{2}, \theta = 0$), the skyrmions have fixed relative isospin orientations $C = 1$ and $C = i\tau_3$, respectively. Along the gradient-flow path originating from two-well separated attractive skyrmions, the $B = 2$ configurations retain their relative isospin orientation, $C = i\tau_2$. Fig. 3 shows these three important paths on the unstable manifold M^{12}. Performing a finite expansion of the potential energy in terms of Wigner-D-functions, as explained in [6], the skyrmion-skyrmion potential can be written approximately as

$$V(R, \beta, \chi) = V_{00}(R) + V_{10}(R)\cos(\beta) + V_{11}(R)\cos^2(\frac{\beta}{2})\cos(\chi), \qquad (5)$$

where the radial coefficients V_{00}, V_{10} and V_{11} are deduced from the three potentials shown in Fig. 3. We expect that the expansion will hold for medium and large distances. At shorter distances, higher-order terms in the expansion, may become important, however.

Inspection of the element $M_{RR}(R, C)$ in the reduced mass tensor along the three im-

portant paths on M^{12}, using the method of [11], shows that this quantity depends rather strongly on the collective coordinates at shorter distances. In order to keep the reduced mass fixed at its asymptotic value $\mu_s = M_s/2$, where $M_s = 1463$ MeV is the single-skyrmion mass in our parameter set, we change the definition of the distance R on M^{12}. Different choices are possible since, a priori, no distance definition is preferred. The only constraint is that all definitions lead to the same result for well separated skyrmions. For the three relevant paths the new distance can be easily calculated [12]. The results for the interaction potential are also shown in Fig. 3. Due to the redefinition of the distance the potential (5) is modified at shorter distances. This indicates further that a detailed knowledge of the metric structure on M^{12} may become of importance for a proper description.

Since a more thorough calculation of the mass tensor $M_{ij}(Q)$ on M^{12} is beyond our means at present, we use its large distance form, in order to define a collective coordinate representation of the dynamics. The approximate Lagrange function in the body-fixed frame then becomes

$$L^{b.f.} = \frac{\mu_s}{2}\dot{R}^2 + \frac{\lambda_s}{2}(\dot{\beta}^2 + \cos^2(\frac{\beta}{2})\dot{\chi}^2) - V(R, \beta, \chi). \tag{6}$$

The quantity $\lambda_s = (265 MeV)^{-1}$ is the moment of inertia of a $B = 1$ skyrmion in our parameter set. The quality of the approximation can be tested by solving the corresponding gradient-flow equations for different initial values of R, β and χ. As a result, we find qualitative agreement of the resulting flow curves with paths of steepest descent on the unstable manifold. Unfortunately, a precise comparison is not possible at the moment, because of the difficulty of a rigorous definition of the relative collective coordinates for all paths on M^{12}.

Treating the nine global collective coordinates in an adiabatic approximation and using the large-distance mass tensor on M^{12}, we arrive – after canonical quantization of all collective coordinates – at a simple Hamiltonian for the $B = 2$ sector. In the c.m. system it reads

$$H = \frac{\hat{p}_R^2}{2\mu_s} + \frac{\vec{L}^2}{2\mu_s R^2} + \frac{1}{2\lambda_s}(\vec{S}_1^2 + \vec{S}_2^2) + 2M_s + V(R, C). \tag{7}$$

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Here $\hat{p}_R$ denotes the radial momentum operator, $\vec{L}$ is the angular momentum operator and $\vec{S}_{1,2}$ are the spin operators of the two interacting baryons. As has been done earlier, we apply a Born-Oppenheimer approximation to the collective coordinate dynamics, in order to deduce a nucleon-nucleon potential from the Hamiltonian above [15]. Taking into account finite-N_c effects [6], the reduced Hamiltonian is diagonalized for given R in the space of asymptotic nucleon and Δ-isobar states. This leads to a nucleon-nucleon potential of the form

$$V_{NN}^{(BO)}(R) = V_c^0(R) + V_c^1(R)(\vec{\tau}_1\vec{\tau}_2) + V_\sigma^0(R)(\vec{\sigma}_1\vec{\sigma}_2) + V_\sigma^1(R)(\vec{\sigma}_1\vec{\sigma}_2)(\vec{\tau}_1\vec{\tau}_2)$$
$$+V_T^0(R)S_{12} + V_T^1(R)S_{12}(\vec{\tau}_1\vec{\tau}_2), \tag{8}$$

where S_{12} denotes the tensor operator. Fig. 4 shows as an example the central part $V_c^0(R)$ in comparison to the phenomenologically successful Argonne potential [16]. The potential deduced from our nuclear Hamiltonian renders almost all features of the realistic potential. It has the correct asymptotic strength, reproduces a medium-range attraction and builds up a repulsive core at shorter distances. This demonstrates that the Skyrme model can provide sizable attraction between nucleons and therefore might serve as a good model for the study of bound states.

In summary, we have presented the first calculation of field configurations on the unstable manifold in the $B = 2$ sector of the Skyrme model. Applying a collective coordinate description, we were able to define a simple nuclear Hamiltonian, approximately valid at medium and large distances. When performing a calculation of the nucleon-nucleon potential within the Born-Oppenheimer approximation, good qualitatively agreement with phenomenological potentials for the central part has been found. To improve on the results, one has to examine the metric structure on the unstable manifold M^{12}, in detail. It has been demonstrated that such a analysis is numerically possible.

ACKNOWLEDGMENTS

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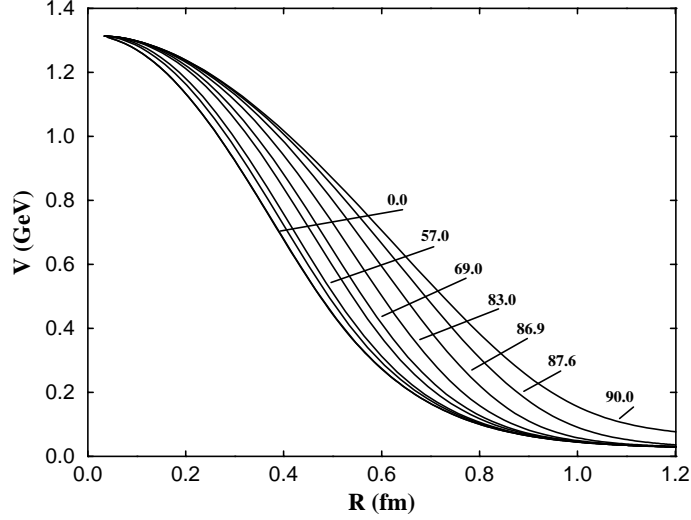[16] R. B. Wiringa, R. A. Smith and T. L. Ainsworth, Phys. Rev.**C29**, 1207 (1984).

FIG. 1. The interaction potential of two skyrmions as a function of the quadrupole distance R along paths of steepest descent from the $B = 2$ hedgehog. Shown are paths resulting from a 'parallel superposition' of the unstable modes ($\theta = 0$), using different mixing angles φ (in degree).
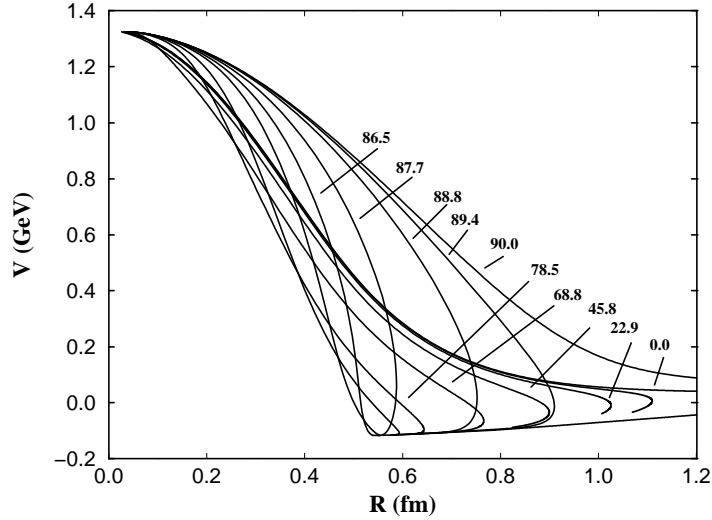

FIG. 2. A different set of paths of steepest descent from the $B = 2$ hedgehog. Shown are paths resulting from an 'orthogonal superposition' of the unstable modes ($\theta = 90$ degree), using different mixing angles φ (in degree).
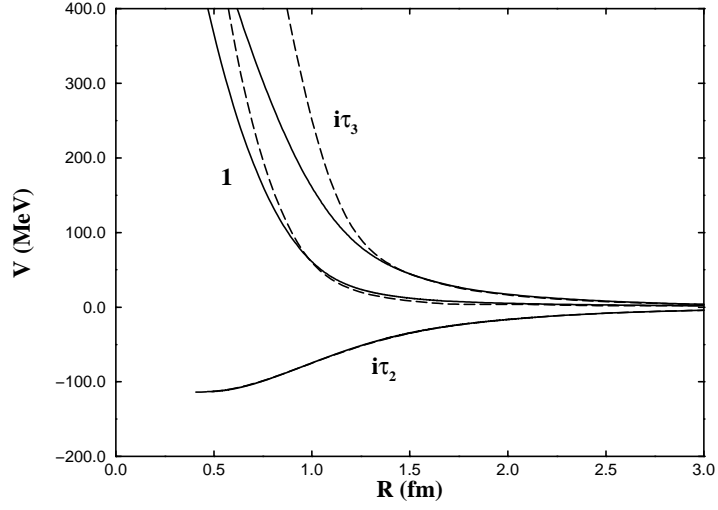
FIG. 3. The skyrmion-skyrmion potential as a function of the distance R and the relative isospin orientation C. The dashed lines give the results using a quadrupole definition for the distance while the solid lines denote the results using a redefined distance definition, as explained in the text.
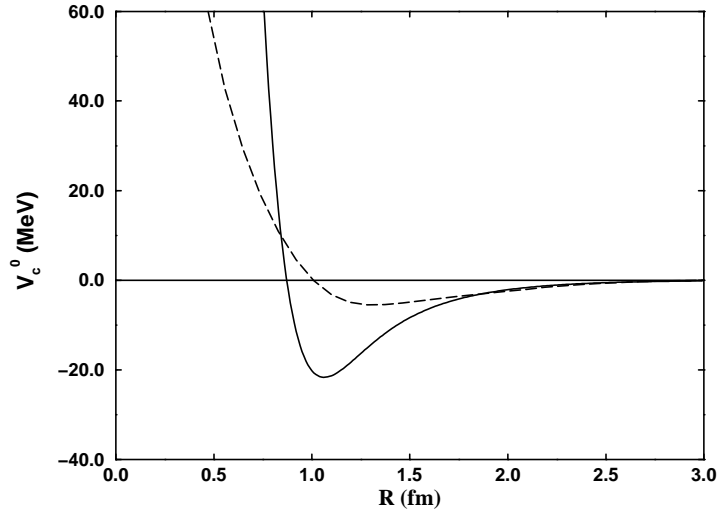
FIG. 4. The central part of the nucleon-nucleon potential for the Argonne potential (solid line) and for the Skyrme model, using the results from the unstable manifold and the Born-Oppenheimer approximation (dashed line).